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NOTES ON BASIC RINGS, II

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Let R be a ring with unit element. In §1 we study the R -endomorphism ring of an R -module. In §2 we consider an algebra with a finite rank over a field K and discuss the connection between an algebra and its basic algebra.

1. Let V be a left R -module. We denote by $E_r(R)$ or simply by $E(R)$ the R -endomorphism ring of V . We consider $E(R)$ as right operator system of V . Evidently $E(R)$ has the unit element $1'$ and V is faithful as a right $E(R)$ -module. If $a \in R$, $a' \in E(R)$, $v \in V$ then

$$(1) \quad a(va') = (av)a'.$$

Let $V = V_1 + V_2$ denote the decomposition of V into a direct sum of left R -modules V_1 and V_2 . If $v = v_1 + v_2$, ($v_1 \in V_1$, $v_2 \in V_2$) then the mapping e' defined by $ve' = v_1$, is an R -endomorphism of V . We see that e' is idempotent and $V_1 = Ve'$, $V_2 = V(1' - e')$:

$$(2) \quad V = Ve' + V(1' - e').$$

Lemma 1. *An idempotent element e' of $E(R)$ is primitive if and only if Ve' is indecomposable as a left R -module.*

Lemma 2. *Let e' and f' be two idempotent elements of $E(R)$. If $e' \cong f'$ then $Ve' \cong Vf'$, and conversely.*

Proof. If $e' \cong f'$ there exist two elements a', b' such that $e' = a'b'$ and $f' = b'a'$ ($a' \in e'E(R)f'$, $b' \in f'E(R)e'$). Hence $Va' = Vf'$. Then $Ve' \cong Vf'$ under the correspondence $ve' \rightarrow va'$. Conversely suppose $Ve' \cong Vf'$ under the correspondence $ve' \rightarrow (ve')^\sigma \in Vf'$. There exist two elements a', b' of $E(R)$ such that

$$\begin{aligned} (ve')a' &= (ve')^\sigma, & (v(1' - e'))a' &= 0, \\ (vf')b' &= (vf')^{\sigma^{-1}}, & (v(1' - f'))b' &= 0. \end{aligned}$$

We then see that $e' = a'b'$ and $f' = b'a'$.

Lemma 3. $E_{V,e'}(R) = e'E(R)e'$ for any idempotent element e' of $E(R)$.

In what follows we assume always that a left R -module V satisfies the following condition:

(**) V is decomposed into a direct sum of a finite number of indecomposable R -modules and this decomposition is unique up to an R -isomorphism.

Let

$$(3) \quad V = \sum_{\kappa=1}^m \sum_{\mu=1}^{\varphi(\kappa)} V_{\kappa, \mu}$$

denote the decomposition of V into a direct sum of indecomposable R -modules $V_{\kappa, \mu}$, where

$$(4) \quad V_{\kappa, 1} \cong V_{\kappa, 2} \cong \dots \cong V_{\kappa, \varphi(\kappa)}$$

and $V_{\kappa, \mu} \not\cong V_{\lambda, \nu}$ for $\kappa \neq \lambda$. Corresponding to the decomposition (3) we have the decomposition of the unit element $1'$ of $E(R)$ into mutually orthogonal primitive idempotent elements:

$$(5) \quad 1' = \sum_{\kappa} \sum_{\mu} e'_{\kappa, \mu}.$$

Here $V_{\kappa, \mu} = Ve'_{\kappa, \mu}$ and

$$(6) \quad V = \sum_{\kappa} \sum_{\mu} Ve'_{\kappa, \mu}.$$

It follows from Lemma 2 that if V satisfies the condition (**) then $E(R)$ satisfies the condition (*) [4, p. 106], and conversely.

We set $e'_{\kappa, 1} = e'_\kappa$, $e' = \sum_{\kappa} e'_\kappa$. $e'E(R)e' = (E(R))^0$ is a basic ring of $E(R)$ and if $\{c'_{\kappa, \mu\nu}\}$ is a system of matrix units corresponding to the decomposition (5) then

$$(7) \quad E(R) = \sum_{\kappa, \lambda} \sum_{\mu, \nu} c'_{\kappa, \mu 1} (E(R))^0 c'_{\lambda, 1 \nu}.$$

The left R -module Ve' is called the reduced module of V . We have by Lemma 3 the following

Lemma 4. $E_{Ve'}(R) = (E(R))^0$.

We assume that V is a faithful left R -module. Denote by $E(E(R))$ the $E(R)$ -endomorphism ring of V (considered as a right $E(R)$ -module). If $a' \in E(R)$, $a'' \in E(E(R))$, $v \in V$ then

$$(8) \quad (a''v)a' = a''(va').$$

Since V is a faithful left R -module it follows from (1) that

$$(9) \quad R \subseteq E(E(R)).$$

Further we see easily that

$$(10) \quad E(R) = E(E(E(R))).$$

We assume hereafter that R satisfies the condition $(*)$ and $e_{i,\alpha}$, e_i , $c_{i,\alpha\beta}$, e , $R^0 = eRe$ have the same meaning as in [4]. Moreover we assume that $1v = v$ for any element v in V . We then have $AV = V$ and

$$(11) \quad V = \sum_{i,\alpha} e_{i,\alpha} V = \sum_{i,\alpha} c_{i,\alpha 1} V,$$

where $e_i V \cong c_{i,\alpha 1} V$ and $c_{i,\alpha 1} V \cong c_{j,\beta 1} V$ for $i \neq j$.

Lemma 5. *If $E(E(R)) = R$ then the right $E(R)$ -modules $e_i V$ ($i = 1, 2, \dots, n$) are indecomposable.*

Lemma 6. $E_{eV}(R^0) = E(R)$.

Proof. The mapping a^* defined by $(ev)a^* = (ev)a'$ ($a' \in E(R)$) is an R^0 -endomorphism of eV since

$$eae((ev)a^*) = eae((ev)a') = (eae v)a' = (eae v)a^*.$$

We have

$$(e_i v)a' = e_i(va') \in e_i V,$$

$$(c_{i,\alpha 1} v)a' = c_{i,\alpha 1}(va') \in c_{i,\alpha 1} V.$$

Hence, if $a' \neq b'$ there exists at least an element $v \in V$ such that $(ev)a' \neq (ev)b'$. This implies $a^* \neq b^*$. Conversely let $a^* \in E_{eV}(R^0)$. We set $(e_i v)a' = (e_i v)a^*$, $(c_{i,\alpha 1} v)a' = c_{i,\alpha 1}((e_i v)a^*)$. We then have

$$\begin{aligned} c_{j,\beta 1} a c_{i,\alpha 1} ((c_{i,\alpha 1} v)a') &= c_{j,\beta 1} a c_{i,\alpha 1} (c_{i,\alpha 1} (e_i v)a^*) \\ &= c_{j,\beta 1} (e_j a e_i) ((e_i v)a^*) = c_{j,\beta 1} ((e_j a e_i v)a^*) \\ &= (c_{j,\beta 1} a e_i v)a' = (c_{j,\beta 1} a c_{i,\alpha 1} c_{i,\alpha 1} v)a'. \end{aligned}$$

Similarly

$$\begin{aligned} c_{j,\beta 1} a c_{i,\alpha 1} ((c_{i,\alpha 1} v)a') &= (c_{j,\beta 1} a c_{i,\alpha 1} (c_{i,\alpha 1} v))a' \\ &= 0 & (i \neq \alpha), \\ c_{j,\beta 1} a c_{k,\alpha 1} ((c_{i,\alpha 1} v)a') &= (c_{j,\beta 1} a c_{k,\alpha 1} (c_{i,\alpha 1} v))a' \\ &= 0 & (k \neq i). \end{aligned}$$

Hence $a' \in E(R)$ and $E_{eV}(R^0) = E(R)$.

Lemmas 4 and 6 are also valid for a right R -module V :

$$(12) \quad E_{eV}(R) = (E(R))^n,$$

$$(13) \quad E_{Ve'}(R^0) = E(R).$$

Theorem 1. $E_{Ve'}(E_{Ve'}(R)) = E(E(R)).$

Proof. Applying (13) to the right $E(R)$ -module V , we have $E_{Ve'}((E(R))^0) = E(E(R)).$ Lemma 4 yields $E_{Ve'}((E(R))^0) = E_{Ve'}(E_{Ve'}(R)).$

Corollary. If $E_{Ve'}(E_{Ve'}(R)) = R$ then $E(E(R)) = R$, and conversely.

Lemma 7. If $E_{eV}(E_{eV}(R^0)) = R^0$ then $E(E(R)) = R$, and conversely.

Proof. By Lemmas 6 and 3

$$\begin{aligned} E_{eV}(E(R)) &= E_{eV}(E_{eV}(R^0)), \\ eE(E(R))e &= E_{eV}(E(R)), \end{aligned}$$

whence $eE(E(R))e = E_{eV}(E_{eV}(R^0)).$ Suppose $E_{eV}(E_{eV}(R^0)) = R^0.$ We then have $eE(E(R))e = R^0$ and

$$\begin{aligned} E(E(R)) &= \sum_{i, \alpha} \sum_{j, \beta} c_{i, \alpha 1} (eE(E(R))e) c_{j, 1 \beta} \\ &= \sum_{i, \alpha} \sum_{j, \beta} c_{i, \alpha 1} R^0 c_{j, 1 \beta} = R. \end{aligned}$$

Conversely if $E(E(R)) = R$ then $eRe = R^0 = E_{eV}(E_{eV}(R^0)).$

Theorem 2. $E_{V_0}(R^0) = (E(R))^0$, where $V_0 = eVe'.$

Proof. Applying Lemma 4 to the left R^0 -module eV , we have $E_{V_0}(R^0) = (E_{eV}(R^0))^0$, whence $E_{V_0}(R^0) = (E(R))^0$ by Lemma 6.

Theorem 3. If $E_{V_0}(E_{V_0}(R^0)) = R^0$ then $E(E(R)) = R$, and conversely.

Proof. Suppose $E_{V_0}(E_{V_0}(R^0)) = R^0.$ By Corollary to Theorem 1, $E_{eV}(E_{eV}(R^0)) = R^0$ and hence $E(E(R)) = R$ by Lemma 7. The converse may be proved easily.

As is well known, if $V \cong R$ (considered as a left R -module) then

$$(14) \quad E_V(E_V(R)) = R.$$

By Corollary to Theorem 1, we obtain

$$(15) \quad E_{Re}(E_{Re}(R)) = R.$$

Generally we have the following

Theorem 4. Let V be a direct sum of any left R -module V_1 and $V_2 = Re.$ Then $E_V(E_V(R)) = R.$

Proof. $eV = eV_1 + eRe = eV_1 + R^0.$ We then have $E_{eV}(E_{eV}(R^0)) = R^0$ by Theorem II - E [3]. Hence $E_V(E_V(R)) = R$ by Lemma 7.

2. We consider always an algebra with a finite rank over a field K and with unit element. Denote by A^0 a basic algebra of an algebra A . Then

$$A = \sum_{i, \alpha} \sum_{j, \beta} c_{i, \alpha 1} A^0 c_{j, 1 \beta}.$$

Let B be a second algebra over the same field K . A and B are called similar if $A^0 \cong B^0$. We write then $A \sim B$. This is a reflexive, symmetric, and transitive relation, by means of which algebras over K are classified into disjoint classes.

Lemma 8. $A \times B \sim A^0 \times B^0$.

Proof. Let $1' = \sum_{j, \beta} e'_{j, \beta}$ denote the decomposition of the unit element $1'$ of B into mutually orthogonal primitive idempotent elements. We set as usual $e'_{j, 1} = e'_j$, $\sum e'_j = e'$. Then $B^0 = e' B e'$. The idempotent elements $e_{i, \alpha} \times e'_{j, \beta}$ of $A \times B$ are not necessarily primitive. Since $e_{i, \alpha} \times e'_{j, \beta} \cong e_i \times e'_j$, $e \times e'$ may be decomposed into two mutually orthogonal idempotent elements e'' and f'' : $e \times e' = e'' + f''$, where e'' is the sum of a maximal system of mutually orthogonal, mutually non-isomorphic primitive idempotent elements of $A \times B$. We then have

$$\begin{aligned} (A \times B)^0 &= e''(A \times B)e'' = e''((e \times e')(A \times B)(e \times e'))e'' \\ &= e''(e A e \times e' B e')e'' = e''(A^0 \times B^0)e'' = (A^0 \times B^0)^0. \end{aligned}$$

Theorem 5. If $A \sim B$, $C \sim D$ then $A \times C \sim B \times D$.

Let A_m denote the complete matrix algebra of degree m with coefficients from A . We see easily that $A_m \sim A$.

Theorem 6. If $A \sim B$ then $A_L \sim B_L$ for any extension field L of K .

Proof. Let $A^0 = e A e$. Evidently $e_i \cong e_{i, \alpha}$ in A_L since $e_i \cong e_{i, \alpha}$ in A . Then $e = e^* + f^*$ ($e^* f^* = f^* e^* = 0$), where e^* is the sum of a maximal system of mutually orthogonal, mutually non-isomorphic primitive idempotent elements of A_L . We have

$$(A_L)^0 = e^* A_L e^* = e^* (e A_L e) e^* = e^* ((A^0)_L) e^* = ((A^0)_L)^0.$$

We consider an A - B -module V such that $1v = v1' = v$ ($v \in V$), where $1'$ denotes the unit element of B . Let $\{c_{i, \alpha \beta}\}$ and $\{c'_{\kappa, \mu \nu}\}$ be the system of matrix units of A and B respectively. Then

$$(16) \quad V = \sum_{i, \alpha} \sum_{\kappa, \nu} c_{i, \alpha 1} V c'_{\kappa, 1 \nu}.$$

Denote by N and N' the radicals of A and B . We set $\bar{A} = A/N$, $\bar{B} = B/N'$. If

$$Ve'_\kappa = V_1e'_\kappa \supset V_2e'_\kappa \supset \cdots \supset V_te'_\kappa \supset 0$$

is a composition series for Ve'_κ considered as a left A -module, then

$$eVe'_\kappa = eV_1e'_\kappa \supset eV_2e'_\kappa \supset \cdots \supset eV_te'_\kappa \supset 0$$

is a composition series for the left A^0 -module eVe'_κ and if $V_ue'_\kappa/V_{u+1}e'_\kappa \cong \bar{A}\bar{e}_i$ then $eV_ue'_\kappa/eV_{u+1}e'_\kappa \cong \bar{e}_i\bar{A}\bar{e}_i$.

Now we assume that V is finite-dimensional over K . Let $h_{i\kappa}$ be the number of the factor groups $\cong \bar{A}\bar{e}_i$ in a composition factor group series of the left A -module Ve'_κ and let $h'_{\kappa i}$ be the number of the factor groups $\cong \bar{e}'_i\bar{B}$ in a composition factor group series of the right B -module e_iV . We can prove in a similar manner as Theorem 3 [1] the following

Theorem 7. $h_{i\kappa}(\text{Rank } \bar{e}_i\bar{A}\bar{e}_i) = h'_{\kappa i}(\text{Rank } \bar{e}'_\kappa\bar{B}\bar{e}'_\kappa)$.

If K is algebraically closed then $h_{i\kappa} = h'_{\kappa i}$. We denote by \mathfrak{A} the representation of A defined by the left A -module V . Let F_i be the irreducible representation of A defined by $\bar{A}\bar{e}_i$ and let U_κ be the representation of A defined by Ve'_κ . Similarly we define \mathfrak{A}' , F'_κ , and U'_i of B . Then

$$(17) \quad U_\kappa \leftrightarrow \sum_i h_{i\kappa} F_i, \quad U'_i \leftrightarrow \sum_\kappa h'_{\kappa i} F'_\kappa,$$

where the sign \leftrightarrow indicates that we have the same irreducible constituents on both sides and so

$$\mathfrak{A} \leftrightarrow \sum_\kappa \sum_i f'(\kappa) h_{i\kappa} F_i, \quad \mathfrak{A}' \leftrightarrow \sum_i \sum_\kappa f(i) h'_{\kappa i} F'_\kappa.$$

This shows that the multiplicity of F_i in \mathfrak{A} is the degree of U'_i and the multiplicity of F'_κ in \mathfrak{A}' is the degree of U_κ .

If V is a left A -module then V may be considered as an $A \cdot E(A)$ -module. Hence above arguments are valid for the $A \cdot E(A)$ -module V .

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